CALCULATION OF EFFECTIVE ELASTIC MODULI OF COMPOSITE MATERIALS WITH MULTIPHASE INTERACTIONS TAKEN INTO CONSIDERATION

A. G. Fokin and T. D. Shermergor

The effective tensors of elastic moduli and pliability were calculated by renormalizing the equations of equilibrium and incompatibility. All multiphase interactions were taken into consideration, but the calculation was done approximately by using only the singular parts of derivatives of Green's functions, which is equivalent to the assumption of uniformity of the random components of stress and strain fields within grain boundaries. Solutions derived by two methods are reconciled, making it possible to determine the position of the elastic moduli of shear and of bulk K between the Hashin-Shtrikman bounds. Particular cases in which only the bulk modulus is not uniform, as well as those of composite materials with all of their constituents having the same ratio $K = 4\mu/3$ are considered.

Deformation of inhomogeneous solid bodies is accompanied by the appearance in the material of an elastic field whose random component is not generally uniform within the boundaries of a grain. Hence, the relationship between the random and the regular components of the field is nonlocal and is defined by a certain set of integral operators. This nonlocalization considerably complicates the problem of calculating the effective elastic constants and the multipoint moments of the elastic field. Only in certain particular cases a quasi-uniform deformation produces a uniform elastic field within the boundaries of a grain. This occurs with respect to the bulk component of the elastic field of a material for which only the bulk modulus is nonuniform [1] and, also, when laminated structures are subject to strain [2]. The limited number of models of solid media with uniform stresses and strains within the boundaries of a grain has led to the development of approximate methods of computation of elastic moduli and of correlation functions for the elastic field, such as the method of self-consistency [3, 4], the theory of random functions [1, 2, 5–10], and others.

Although the renormalization of equations of equilibrium and of incompatibility permit in principle an exact determination of effective moduli of elasticity, the mathematical difficulties arising from the presence of nonlocal bonds make it impossible to advance beyond the third approximation [7]. If, however, an approximation to localization is made, it becomes possible to sum all terms of the series. Summation of series in the framework of the perturbations theory was first used in the calculation of elastic moduli by Bolotin and Moskalenko [11, 12], who had introduced for this purpose the concept of highly isotropic polycrystals. Kröner had used a similar approach in the investigation of elastic properties of polycrystals [13].

The method of renormalization in an approximation to localization applicable to composite materials is developed in the following. All multiphase interactions are taken into consideration in the calculation of the effective moduli of elasticity, except that certain approximate expressions are substituted for the exact derivatives of the Green's functions in the equations of equilibrium and incompatibility, which is equivalent to the transition within grain boundaries from a nonuniform to a uniform field. The principle of reconciliation of solutions [3, 14] obtained by various methods permits the determination of the bulk and shear moduli of elasticity within the Hashin-Shtrikman bounds.

1. Let us express the random components of tensors of stress σ_{ij} and strain ε_{ij} in terms of their regular values. For an arbitrary inhomogeneous linear medium, generally containing sources of internal stresses, the equations of equilibrium and of incompatibility may be, respectively, written as:

$$L_{il}u_l + f_i = 0, \quad L_{il} \equiv \nabla_k \lambda_{iklm} \nabla_m, \quad \lambda_{iklm} = K\delta_{ik}\delta_{lm} + \mu D_{iklm},$$ (1.1)

and

$$L_{iklm}\sigma_{lm} + \eta_{ik} = 0, \quad L_{iklm} \equiv \mathrm{Rot}_{ikpq}s_{pqlm},$$
$$s_{pqlm} = p\delta_{pq}\delta_{lm} + qD_{pqlm}, \quad p = \frac{1}{9K}, \quad q = \frac{1}{4\mu},$$
$$D_{iklm} = \delta_{il}\delta_{km} + \delta_{im}\delta_{kl} - \tfrac{2}{3}\delta_{ik}\delta_{lm}, \quad \mathrm{Rot}_{ikpq} \equiv e_{ijp}e_{knq}\nabla_j\nabla_n.$$ (1.2)

Here u_l is the translation vector, f_i is the volume density of external forces, η_{ik} is the incompatibility tensor, e_{ijp} is a unit skew-symmetric tensor, λ_{iklm} and s_{pqlm} are the tensors of elastic moduli and of pliability, and K and μ the bulk and shear moduli, respectively.

By decomposing the operators and functions into their regular and random components it is possible to obtain the following relationships:

$$u_i{}' = -(M_{il}L_{lm}{}^\circ + \delta_{im})\langle u_m \rangle, \tag{1.3}$$

$$\sigma_{ik}' = -(M_{iklm}L_{lmpq}^\circ + \delta_{i(p}\delta_{q)k})\langle \sigma_{pq} \rangle. \tag{1.4}$$

The operators M and L° are defined by the matrix equalities

$$LM + I = 0, \quad L^\circ \langle M \rangle + I = 0, \tag{1.5}$$

with the unit matrix I having δ_{ij} as its elements in the case of second rank and $\delta_{i(p}\delta_{q)k}$ in that of rank four. Random and regular components are denoted, respectively, by primes and angle brackets. The averaging is carried out over regions whose dimensions exceed the spatial correlation scale but are small in comparison with the distances over which regular components of operators and functions vary substantially.

Introducing operators X and M°,

$$X = M^\circ L', \quad \langle L \rangle M^\circ + I = 0, \tag{1.6}$$

we can express operators M and L° in the form

$$M = \sum_0^\infty X^n M^\circ, \quad L^\circ = \langle L \rangle \sum_0^\infty \left(-\sum_1^\infty \langle X^k \rangle\right)^n. \tag{1.7}$$

Expanding the operational series we obtain

$$Z' = R\langle Z \rangle, \quad R \equiv \sum_1^\infty Q_n', \tag{1.8}$$

$$Q_n = X^n - \sum_{k=1}^{n-1} X^{n-k}\langle X^k \rangle + \sum_{k,\,l=1}^{k+l \leqslant n-1} X^{n-k-l}\langle X^k \rangle \langle X^l \rangle - \ldots \tag{1.9}$$

Here Z is u_i or σ_{ij} in matrix notation. The integral operator M° is defined by the regular Green's function [8]

$$M^\circ f = G * f = \int G(\mathbf{r} - \rho) f(\rho) d\rho. \tag{1.10}$$

2. Let us apply the above method to the calculation of effective elastic moduli. Within a single grain the transition from a nonuniform to a uniform field can be effected with only the singular parts of the derivatives of the Green's functions taken into account. This makes it possible to reduce the operator expansion defined by formulas (1.8) and (1.9) to a numerical one, which is equivalent to the approximation to local bond.

Let us first renormalize the equations of equilibrium, selecting operator (1.1) as L. From expression (1.8) we then obtain

$$u_i' = R_{ik}\langle u_k \rangle, \quad \varepsilon_{ij}' = H_{ijkl}\langle \varepsilon_{kl} \rangle. \tag{2.1}$$

Here $\varepsilon_{ij} = u_{(ij)}$ is the strain tensor, and operators H_{ijkl} and R_{ik} are bound by the obvious equality

$$\nabla_j R_{ik}\langle u_k \rangle = H_{ijkl}\langle \nabla_l u_k \rangle. \tag{2.2}$$

Let us confine our analysis to uniform macrostrains $\langle \varepsilon_{kl} \rangle$ = const. The second derivative of the Green's function is [15]

$$G_{ij,\,kl} = G^{(s)}_{ij,\,kl} + G^{(f)}_{ij,\,kl}; \quad G^{(s)}_{ij,\,kl} = -\frac{\delta(\mathbf{r})}{3\langle\mu\rangle}\left(\delta_{ij}\delta_{kl} - \frac{\varkappa}{5}\delta_{ijkl}\right),$$

$$G^{(f)}_{ij,\,kl} = -\frac{1}{8\pi\langle\mu\rangle r^3}[2\delta_{ij}(\delta_{kl} - 3\psi_{kl}) - \varkappa(\delta_{ijkl} - 3\varphi_{ijkl} + 15\psi_{ijkl})]; \tag{2.3}$$

$$\varkappa = \frac{\langle 3K + \mu \rangle}{\langle 3K + 4\mu \rangle}, \quad \delta_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk},$$

$$\varphi_{ijkl} = \delta_{ij}\psi_{kl} + \delta_{kl}\psi_{ij} + \delta_{il}\psi_{jk} + \delta_{jk}\psi_{il} + \delta_{ik}\psi_{jl} + \delta_{jl}\psi_{ik},$$

$$\psi_{ijkl} = n_i n_j n_k n_l, \quad \psi_{ij} = n_i n_j, \quad n_i = x_i r^{-1}. \tag{2.4}$$

Here $G^{(s)}_{ij,kl}$ and $G^{(f)}_{ij,kl}$ define, respectively, the singular and the formal parts of the derivative. The presence of the co-factor $\delta(r)$ in the singular part of the derivative transforms the integral operator $G^{(s)}_{ij,kl}*$ into the constant tensor

$$G^{(s)}_{ij,kl} * \lambda'_{jlmn} = - ({}^1\!/_3 K_0' \delta_{ik}\delta_{mn} + {}^1\!/_2 \mu_0' D_{ikmn}) \equiv J_{ikmn},$$

$$K_0' \equiv \frac{3K'}{\langle 3K + 4\mu \rangle}, \quad \mu_0' \equiv \frac{6\mu' \langle K + 2\mu \rangle}{5 \langle \mu \rangle \langle 3K + 4\mu \rangle}. \tag{2.5}$$

Thus, the singular part $G^{(s)}_{ij,kl}*$ defines localized bonds, while the formal derivative $G^{(f)}_{ij,kl}*$, which is an integral operator, takes into account the deviation of strain within the boundaries of a grain from its average value across the latter. Hence, in the approximation of local bond between components of the elastic field it is possible to disregard the terms $G^{(f)}_{ij,kl}*$, and assume that $G_{ij,kl} \approx G^{(s)}_{ij,kl}$.

Taking into account the relationships

$$J_{ijpq} J_{pqkl} = {}^1\!/_3 K_0'^2 \delta_{ij}\delta_{kl} + {}^1\!/_2 \mu_0'^2 D_{ijkl},$$

$$K_0'^2 - \langle K_0'^2 \rangle = - K_0' \xi, \quad \mu_0'^2 - \langle \mu_0'^2 \rangle = - \mu_0' \eta,$$

$$\xi \equiv \frac{3 (c_1 - c_2)(K_1 - K_2)}{\langle 3K + 4\mu \rangle}, \quad \eta \equiv \frac{6 (c_1 - c_2)(\mu_1 - \mu_2) \langle K + 2\mu \rangle}{5 \langle \mu \rangle \langle 3K + 4\mu \rangle}. \tag{2.6}$$

from equalities (1.8), (1.9), (2.1), (2.2), (2.4), and (2.6) we obtain

$$\varepsilon_{ij}' = - \left({}^1\!/_3 K_0' \sum_0^\infty \xi^n \delta_{ij}\delta_{kl} + {}^1\!/_2 \mu_0' \sum_0^\infty \eta^n D_{ijkl} \right) \langle \varepsilon_{kl} \rangle. \tag{2.7}$$

Expression (2.7) makes it possible to write in explicit form the expression for tensor H_{ijkl} which defines the relation between total stresses and strains at any arbitrary point of a material whose regular component of the stress tensor is

$$\varepsilon_{ij} = (I_{ijkl} + H_{ijkl}) \langle \varepsilon_{kl} \rangle, \quad I_{ijkl} = \delta_{i(k}\delta_{l)j},$$

$$- H_{ijkl} = {}^1\!/_3 K_0' (1 - \xi)^{-1} \delta_{ij}\delta_{kl} + {}^1\!/_2 \mu_0' (1 - \eta)^{-1} D_{ijkl}, \quad \sigma_{ij} = \Lambda_{ijkl} \langle \varepsilon_{kl} \rangle, \tag{2.8}$$

$$\Lambda_{ijkl} = K \left(1 - \frac{K_0'}{1 - \xi} \right) \delta_{ij}\delta_{kl} + \mu \left(1 - \frac{\mu_0'}{1 - \eta} \right) D_{ijkl}. \tag{2.9}$$

Averaging expression (2.9), we obtain

$$\Lambda^u_{ijkl} = \langle \Lambda_{ijkl} \rangle = K^u \delta_{ij}\delta_{kl} + \mu^u D_{ijkl},$$

$$K^u = \langle K \rangle - D_K (c_1 K_2 + c_2 K_1 + a)^{-1}; \tag{2.10}$$

$$\mu^u = \langle \mu \rangle - D_\mu (c_1 \mu_2 + c_2 \mu_1 + b)^{-1},$$

$$D_K \equiv \langle K'^2 \rangle, \quad D_\mu \equiv \langle \mu'^2 \rangle, \quad a \equiv {}^4\!/_3 \mu, \quad b = {}^1\!/_6 \langle \mu \rangle \langle 9K + 8\mu \rangle \langle K + 2\mu \rangle^{-1}. \tag{2.11}$$

Expression (2.3) shows that the averaging of the formal component of the Green's function over angles yields zero. This makes it possible to obtain an approximation to localization by another method—that of disregarding the angular relationships of the two-point mixed correlation functions of elastic moduli and of strains. Such an approach [10] also yields relationships (2.11).

3. Let us now renormalize the incompatibility equation in the approximation of local bonds between field components. Choosing operator (1.2) for L in (1.8), for the random component of the stress tensor we obtain

$$\sigma_{ij}' = R_{ijkl} \langle \sigma_{kl} \rangle. \tag{3.1}$$

The operator X_{ijkl} in the expression of the random operator R_{ijkl} may be written as

$$X_{ijkl} = G^{(s)}_{ijpq} * \mathrm{Rot}_{pqrs} s'_{rskl} =$$

$$= - \frac{1}{3 \langle q \rangle \langle 3p + q \rangle} [\langle q \rangle \delta_{ij} \delta_{rs} + {}^9/_{20} \langle 2p + q \rangle D_{ijrs}] s'_{rskl}. \tag{3.2}$$

Substituting here the explicit expression of the random component of the pliability tensor in accordance with (1.2), we obtain

$$X_{ijkl} = - ({}^1/_3 p_0' \delta_{ij} \delta_{kl} + {}^1/_2 q_0' D_{ijkl}),$$

$$p_0' \equiv \frac{3p'}{\langle 3p + q \rangle}, \quad q_0' \equiv \frac{3q' \langle 2p + q \rangle}{5 \langle q \rangle \langle 3p + q \rangle}. \tag{3.3}$$

Only the singular parts of the derivatives of the Green's functions were taken into consideration in the calculation of X_{ijkl}. The decrease of the Green's function at infinity and the operator relationship $\delta(r)* = 1$ were also taken into account. Thus, the neglect of the formal components of the derivatives of the Green's functions results in the degeneration of the integral operator X_{ijkl} into a random tensor function, which is equivalent to a change within grain boundaries from a nonuniform to a uniform elastic field.

Taking into account that the product of tensor functions X_{ijkl} is

$$X_{ijpq} X_{pqkl} = {}^1/_3 p_0'^2 \delta_{ij} \delta_{kl} + {}^1/_2 q_0'^2 D_{ijkl}. \tag{3.4}$$

and that

$$(Q_n')_{ijkl} = - {}^1/_3 p_0' \alpha^{n-1} \delta_{ij} \delta_{kl} - {}^1/_2 q_0' \beta^{n-1} D_{ijkl}, \tag{3.5}$$

$$\alpha \equiv \frac{3 (c_1 - c_2)(p_1 - p_2)}{\langle 3p + q \rangle}, \quad \beta \equiv \frac{3 (c_1 - c_2)(q_1 - q_2) \langle 2p + q \rangle}{5 \langle q \rangle \langle 3p + q \rangle}. \tag{3.6}$$

from equalities (1.8), (3.1), and (3.5) we find

$$- R_{ijkl} = {}^1/_3 p_0' (1 - \alpha)^{-1} \delta_{ij} \delta_{kl} + {}^1/_2 q_0' (1 - \beta)^{-1} D_{ijkl}, \tag{3.7}$$

$$\sigma_{ij} = (I_{ijkl} + R_{ijkl}) \langle \sigma_{kl} \rangle, \quad \varepsilon_{ij} = S_{ijkl} \langle \sigma_{kl} \rangle, \tag{3.8}$$

$$S_{ijkl} = p \left(1 - \frac{p_0'}{1 - \alpha}\right) \delta_{ij} \delta_{kl} + q \left(1 - \frac{q_0'}{1 - \beta}\right) D_{ijkl}. \tag{3.9}$$

After averaging, we obtain from this the upper value of the elastic pliability tensor

$$S^u_{ijkl} = \langle S_{ijkl} \rangle = p^u \delta_{ij} \delta_{kl} + q^u D_{ijkl}, \tag{3.10}$$

$$9 p^u = \left\langle \frac{1}{K} \right\rangle - D_{1/K} \left[\frac{c_1}{K_2} + \frac{c_2}{K_1} + \left\langle \frac{3}{4\mu} \right\rangle \right]^{-1} \equiv \frac{1}{K^l},$$

$$4 q^u = \left\langle \frac{1}{\mu} \right\rangle - D_{1/\mu} \left[\frac{c_1}{\mu_2} + \frac{c_2}{\mu_1} + \left\langle \frac{6}{\mu} \right\rangle \left\langle \frac{1}{\mu} + \frac{2}{K} \right\rangle \left\langle \frac{8}{K} + \frac{9}{\mu} \right\rangle^{-1} \right]^{-1} \equiv \frac{1}{\mu^l}. \tag{3.11}$$

Here K^l and μ^l are the lower values of the bulk and shear moduli, respectively.

Let us compare the derivation of the effective tensors of elastic moduli by the renormalization of the equations of equilibrium and incompatibility. In the approximation of localized random fields it is possible to present both solutions in the form of expansions. However, since $\alpha \xi \leq 0$ and $\beta \eta \leq 0$, one of these methods yields a fixed-sign and the other an alternating expansion. Thus, when the concentration of the constituent of higher moduli of elasticity exceeds that of the second constituent, the renormalization of the equilibrium equation results in an alternating expansion, and $\xi > 0$ and $\eta > 0$. Otherwise a fixed-sign expansion is obtained when the incompatibility equations are renormalized.

4. Formulas (2.11) and (3.11) provide approximate values of the effective bulk and shear moduli of elasticity. The approximation of localized bond between the random and the regular components of the elastic field was used in the derivation of both formulas. It follows from this that, in the case of an inhomogeneous material in which the random stress and strain tensors are uniform within the boundaries of a grain, these formulas exactly define the effective moduli of elasticity. Formulas (2.11) and (3.11) are also valid in the absence of localized bonds, provided however, that the nature of the nonlocalized bonds is not affected when multiphase interactions are allowed for, i.e., when account for such interactions does not require the introduction of multipoint moments of elastic moduli. The simplest model of this kind is that of a material whose bulk modulus only is nonuniform. In fact, from equalities (1.8)–(1.10) and (2.1)–(2.4) we find that the random component of the strain tensor in first and second approximations

are

$$\varepsilon_{ik}^{\prime(N)} = G_{ij,jk} * \Theta^{(N)} \langle \varepsilon_{ll} \rangle ,$$

$$G_{ij,jk} = -\frac{11}{\langle 3K + 4\mu \rangle} \left[\delta(r) \delta_{ik} + \frac{3}{4\pi r^3} (\delta_{ik} - 3\psi_{ik}) \right],$$

$$\Theta^{(1)} = K', \quad \Theta^{(2)} = K'[1 - K' \langle K + {}^4/_3 \mu \rangle^{-1}]. \tag{4.1}$$

For a material with a nonuniform bulk modulus but with the same shear moduli of its phases, both formulas (2.11) and (3.11) yield the same results [1, 16]:

$$K^u = \langle K \rangle - \frac{D_K}{c_1 K_2 + c_2 K_1 + {}^4/_3 \mu},$$

$$\frac{1}{K^l} = \left\langle \frac{1}{K} \right\rangle - D_{1/K} \left(\frac{c_1}{K_2} + \frac{c_2}{K_1} + \frac{3}{4\mu} \right)^{-1} = \frac{1}{K^u}. \tag{4.2}$$

When both the K and the μ moduli are nonuniform, their bounds are, according to Hashin and Shtrikman [17], defined by the following expressions:

$$K_{\pm} = \langle K \rangle - \frac{D_K}{c_1 K_2 + c_2 K_1 + a_{\pm}}, \quad \mu_{\pm} = \langle \mu \rangle - \frac{D_{\mu}}{c_1 \mu_2 + c_2 \mu_1 + b_{\pm}}. \tag{4.3}$$

Parameters $a_{\pm}$ and $b_{\pm}$ are defined by equalities (2.11) in which the elastic moduli of the first (subscript plus) and of the second (subscript minus) constitutents are substituted for $\langle K \rangle$ and $\langle \mu \rangle$ on the assumption that $K_1 > K_2$ and $\mu_1 > \mu_2$. Similarly, the Hashin values of $K_{\pm}$ and $\mu_{\pm}$ may be derived for pliability from formulas (3.11) by a corresponding substitution in the correlation corrections to the first- and second-phase pliabilities for the mean values $\langle 1/K \rangle$ and $\langle 1/\mu \rangle$ of these.

It follows from the above that relationships (2.11) and (3.11) can also be obtained by the Hashin-Shtrikman method by a formal substitution of $\langle \mu \rangle$ and $\langle 1/\mu \rangle^{-1}$ for μ_i, and similarly for K. It appears that in this case $\mu^l - \mu^u < \mu_+ - \mu$. This apparently was first pointed out by Aleksandrov [14] while analyzing the problem of narrowing the bounds of elastic moduli of cubic polycrystals.

5. Depending on the relation between the parameters of a blend its true elastic moduli may lie either in the middle of their bounds or close to the latter [18]. It is interesting, in this context, to develop an algorithm which would make it possible to indicate approximately the position of the elastic modulus within known bounds. For this we shall require the coincidence of the two expressions (2.11) and (3.11). The necessary and sufficient condition for such coincidence is for the second terms in formulas (2.11) and (3.11) to satisfy the following relationships:

$$\langle K \rangle = \langle 1/K \rangle^{-1}, \quad \langle \mu \rangle = \langle 1/\mu \rangle^{-1}. \tag{5.1}$$

Relationships (5.1) are satisfied in two cases: when the elastic moduli coincide (it was this trivial case which was used by Hashin and Shtrikman for establishing the bounds), and when the moduli are the effective and not the average ones. Let us assume the latter, and substitute into the second terms of (2.11) and (3.11) the effective values of elastic moduli for their mean values. We then obtain in the two cases the same formulas

$$K^* = \langle K \rangle - D_K (c_1 K_2 + c_2 K_1 + {}^4/_3 \mu^*)^{-1},$$

$$\mu^* = \langle \mu \rangle - D_{\mu} \left[c_1 \mu_2 + c_2 \mu_1 + \frac{\mu^* (9K^* + 8\mu^*)}{6 (K^* + 2\mu^*)} \right]^{-1}. \tag{5.2}$$

Relationships (5.2) represent a system of two equations in two unknowns, whose solution yields the sought moduli of elasticity. It is readily seen that from the inequalities

$$K_2 \leqslant \langle 1/K \rangle^{-1} \leqslant K^* \leqslant \langle K \rangle \leqslant K_1, \quad \mu_2 \leqslant \langle 1/\mu \rangle^{-1} \leqslant \mu^* \leqslant \langle \mu \rangle \leqslant \mu_1 \tag{5.3}$$

follows

$$K_- \leqslant K^l \leqslant K^* \leqslant K^u \leqslant K_+, \quad \mu_- \leqslant \mu^l \leqslant \mu^* \leqslant \mu^u \leqslant \mu_+. \tag{5.4}$$

Although K_+, K_-, μ_+, μ_- have been determined for $(K_1 - K_2)(\mu_1 - \mu_2) > 0$, the bounds for K and μ, determined by formulas (2.11) and (3.11), are valid for any sign of the product of differences of moduli.

As an example, let us consider a mechanical mixture whose every constituent is such that the relation $K = 4\mu/3$ is satisfied. In this case, equalities (5.2) reduce to a quadratic equation whose solution is of the form

$$K^* = {}^4/_3\mu^* = {}^1/_2(c_1 - c_2)\ (K_1 - K_2) + [K_1 K_2 + {}^1/_4\ (c_1 - c_2)^2(K_1 - K_2)^2]^{1/2}. \tag{5.5}$$

If $c_1 = c_2$, expression (5.5) results in a geometric mean which coincides with the known empirical rule of Lichtenacker [19, 20],

$$\ln K^* = c_1 \ln K_1 + c_2 \ln K_2 = \ln\sqrt{K_1 K_2}. \tag{5.6}$$

Equalities (5.2) were derived with the use of the theory of random functions without, however, taking into account the extent of the bond between constituents. Hence the obtained results must be applied with caution to matrix mixtures in which one of the constituents occupies a simply connected region and the second a multiply connected one. This applies, in particular, to porous media. Thus, putting $K_2 = \mu_2 = 0$, from Eqs. (5.2) we obtain

$$8x^2 + [3\gamma(2 + c_1) + 4(3-5c_1)]x + 9(1 - 2c_1)\gamma = 0,$$
$$x \equiv \mu^*/\mu, \quad \gamma \equiv K_1/\mu_1. \tag{5.7}$$

and, also, the trivial roots $\mu^* = K^* = 0$, which indicates a possible breakdown of the material cohesion.

It will be seen from Eq. (5.7) that there are no positive roots when $c_1 < 1/2$, while for $c_1 > 1\ 2$ the equation admits positive solutions. For example, for $\gamma = 4/3$ we have $x = 1 - 2c_2$, which coincides with the results of calculation by the method of virial expansion in the approximation to linear concentration.

REFERENCES

1. A. G. Fokin and T. D. Shermergor, "Calculation of elastic moduli of heterogeneous media," PMTF [Journal of Applied Mechanics and Technical Physics], vol. 9, no. 3, 1968.
2. A. G. Fokin and T. D. Shermergor, "Statistical definition of the elastic field of laminated materials," Inzh. zh. MTT [Mechanics of Solids], no. 4, 1968.
3. A. V. Hershey, "The elasticity of an isotropic aggregate of anisotropic cubic crystals," J. Appl. Mech., vol. 21, no. 3, 1954.
4. R. Hill, "A self-consistent mechanics of composite materials," J. Mech. Phys. Solids, vol. 13, no. 4, pp. 213–222, 1965.
5. I. M. Lifshits and L. N. Rozentsveig, "The theory of elastic properties of polycrystals," ZhETF, vol. 16, no. 11, 1946.
6. B. M. Darinskii and T. D. Shermergor, "Elastic moduli of cubic polycrystals," PMTF [Journal of Applied Mechanics and Technical Physics], vol. 6, no. 4, 1965.
7. A. G. Fokin and T. D. Shermergor, "The calculation of bounds of elastic moduli of homogeneous quasi-isotropic media," collection: Application of Mathematical Methods in Geology [in Russian], Izd-vo Nauka, KazakhSSR, Alma-Ata, 1968.
8. B. M. Darinskii, A. G. Fokin, and T. D. Shermergor, "Calculation of elastic moduli of polycrystals," PMTF [Journal of Applied Mechanics and Technical Physics], vol. 8, no. 5, 1967.
9. S. D. Volkov and N. A. Klinskikh, "The theory of elastic properties of polycrystals," Fiz. metallov i metallovedenie, vol. 19, no. 1, 1965.
10. L. P. Khoroshun, "The theory of isotropic strain in bodies with random inhomogeneities," Prikl. mekh. [Soviet Applied Mechanics], vol. 3, no. 9, 1967.
11. V. V. Bolotin and V. N. Moskalenko, "The problem of determination of elastic constants of a microheterogeneous medium," PMTF [Journal of Applied Mechanics and Technical Physics], vol. 9, no. 1, 1968.
12. V. V. Bolotin and V. N. Moskalenko, "Macroscopic properties of microheterogeneous solid bodies," Dokl. AN SSSR, vol. 178, no. 3, 1968.
13. E. Kröner, "Elastic moduli of perfectly disordered composite materials," J. Mech. Phys. Solids, vol. 15, no. 4, 1967.
14. K. S. Aleksandrov, "Calculation of elastic constants of quasi-isotropic polycrystallic materials," Dokl. AN SSSR, vol. 176, no. 2, 1967.
15. I. M. Gelfand and G. E. Shilov, Generalized Functions and Operations on Them [in Russian], 1-st edition, Fizmatgiz, Moscow, 1959.
16. R. Hill, "The elastic properties of composite materials: certain theoretical principles," Collection: Mekhanika [Russian translation], no. 5, 1964.

17. Z. Hashin and S. Shtrikman, "A variational approach to the elastic behavior of multiphase materials," J. Mech. Phys. Solids, vol. 11, no. 2, 1963.

18. A. G. Fokin and T. D. Shermergor, "Calculation of elastic moduli of inhomogeneous materials," Mekh. polimerov [Polymer Mechanics], vol. 4, no. 4, 1968.

19. K. S. Aleksandrov and L. A. Aizenberg, "Methods of calculating physical constants of polycrystallic materials," Dokl. AN SSSR, vol. 167, no. 5, 1966.

20. A. R. Hippel, Dielectrics and Waves [Russian translation], Izd-vo inostr. lit., 1960.

21. M. A. Krivoglaz and A. S. Chervenko, "The elastic moduli of mixtures," Fiz. metallov i metallovedenie, vol. 8, no. 2, 1959.

4 March 1968

Moscow